A HYBRID METHOD FOR PARAXIAL BEAM PROPAGATION IN MULTIMODE WAVEGUIDES

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ABSTRACT

A hybrid (non-ray, non-modal) method for computing the fields of a paraxial beam propagating in a multimode waveguide (parallel-plate or dielectric slab) at large axial distances is presented. The method is based on the Fourier and Fresnel self-imaging properties of these waveguides, and is capable of high accuracy. The method is much more efficient than ray or mode approaches, while giving complete field information which coupled-power equations do not provide.

Introduction

Multimode optical fibers appear at present to be the most common optical waveguiding medium for applications in the immediate future. Incoherent sources and relatively simple detectors can be used, and the tolerance problems encountered with single-mode fibers are far less severe with such waveguides.

At present, there are essentially three methods available for field computation in multimode waveguides. First, one can take a pure modal approach--the excitation amplitude of each mode is computed, and all modes are summed together. Although in principle exact, this approach suffers not only from the large number of modes which must be kept track of (100 - 1000 for a typical fiber; 30 - 100 for a slab geometry) but also from a large degree of cancellation of terms in the mode sum when the field does not match that of an individual mode. Although in some special cases approximate closed-form results are available, a computer analysis is generally required, and roundoff errors can be expected to accumulate, especially for large propagation distances.

A second approach is that of geometrical optics (sometimes encountered as the WKB method). An excellent discussion of this approach has been given by Gloge and Marcatili [1]. Here one approximates the effect of a large number of discrete propagating modes by a continuously distributed propagation constant belonging to a "continuous spectrum" of modes. The propagation problem then reduces to that of determining the amplitude with which each of a cone of rays is excited, and tracing it down the length of the guide. In a situation where paraxial propagation conditions exist (see below), a large number of rays can be expected to contribute at large propagation distances (hundreds of meters or several kilometers may not be uncommon). In this region, the geometrical optics approach can be seen to suffer from similar disadvantages as does the first.

A third approach is a purely numerical one, wherein the partial differential equation is tackled directly, without the use of either mode or ray concepts. This method, like the first, is also capable of arbitrary accuracy in principle, and requires neither a detailed knowledge of a large number of modes, nor the tracing of a large number of ray paths. Again, however, when very long propagation distances are being studied, the discretication of the wave equation in the longitudinal direction can lead to large error accumulations which do not seem easily avoidable by this technique.

Finally, we might also mention here the coupled-power equations approach [2]. This method seeks only to find the total power carried by each mode, since for many applications the details of the field distribution from each mode are not of interest. One then takes a statistical approach to these equations, and obtains useful results for pulse dispersion when each mode of the guide is detectable only through its total power. There are many other applications, however, when the fields themselves are important, such as in the design of couplers, splitters, switches, splicers, etc., and it is this problem in which we are interested here.

The method we propose is based on the imaging properties of multimode waveguides. In the paraxial approximation, a parallel-plate or dielectric slab waveguide will periodically reconstruct the field pattern at the input plane (and, at more frequent intervals, a string of such replicas). Because of this, we need only perform our field computations within the space of one of these periods, and will not suffer the loss of accuracy at large distances associated with the methods described above. Our computations will be performed for a parallel plate waveguide with perfectly conducting walls, but the results are immediately applicable to the dielectric slab waveguide. The method will allow a simple formula to be obtained for the propagation of a Gaussian beam of substantially narrower width than that of the guide.

The Paraxial Approximation

To fix ideas, let us consider the parallel-plate waveguide illustrated in Fig.1. The walls at $x=0$ and $x=a$ are perfectly conducting, and some known source produces a given excitation or input field at the plane $z=0$. For simplicity, we restrict ourselves to two-dimensional, TE fields, so that the entire field $\bar{H}=\bar{a}_xH_x+\bar{a}_zH_z$, $\bar{E}=\bar{a}_yE_y$, where $\bar{a}_x,\bar{a}_y,\bar{a}_z$ are Cartesian unit vectors, can be derived from the scalar function E_y which satisfies

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}+k^2\right)E_y = 0 \qquad (1)$$

for $z>0$. Here $k=\omega\sqrt{\mu\varepsilon}$, where a time dependence $\exp(i\omega t)$ has been assumed, and μ,ε are the electrical parameters of the medium filling the waveguide.

In the paraxial approximation, we write

$$E_y(z,x)=e^{-ikz}A(x,z) \qquad (2)$$

and assume that most propagation takes place nearly in the z-direction; that is, $A(x,z)$ as a function of z varies slowly compared to $\exp(-ikz)$. Inserting (2) into (1) we obtain

$$\left(\frac{\partial^2}{\partial x^2}-2ik\frac{\partial}{\partial z}+\frac{\partial^2}{\partial z^2}\right)A(x,z) = 0 \qquad (3)$$

and in the paraxial approximation, we neglect the $\partial^2/\partial z^2$ term compared to the first derivative term because of the slow variation of $A(x,z)$ in z assumed above. We thus obtain the following parabolic equation for $A(x,z)$[3]:

$$\left(\frac{\partial^2}{\partial x^2}-2ik\frac{\partial}{\partial z}\right)A(x,z) = 0 \qquad (4)$$

To put this approximation on a more quantitative footing, we can apply some ideas from the boundary-layer technique [4].

Using these techniques, it can be shown that a criterion for the accuracy of the paraxial approximation is that

$$kz \ll (ka)^4 \tag{5}$$

In addition, of course, we also have the condition $k^2a^2 \gg 1$, which is implicit in the fact that the guide is highly multimode.

It can be shown [5] that a multimode dielectirc slab waveguide of width b, core index n_1 and cladding index n_2 can be replaced by an equivalent parallel-plate waveguide of width $a = b(1 + 2/k_o\sqrt{n_1^2 - n_2^2}b)$ in the paraxial approximation, where k_o is the wavenumber of free space. Our results for the parallel-plate waveguide can then be directly applied to the case of a simple optical waveguide.

Green's Function and Imaging

By well-known techniques, the field $E_y(x,z)$ for $z > 0$ in the waveguide can be expressed in terms of the field $E_y(x,0)$ at an input plane ($z = 0$) by means of a Green's function $G(x,x';z)$:

$$E_y(x,z) = \int_0^a E_y(x',0)G(x,x';z)dx' \tag{6}$$

The Green's function G_0 for the paraxial approximation (2),(4) to E_y is given by:

$$G_0(x,x';z) = \frac{2}{a} e^{-ikz} \sum_{m=1}^{\infty} \sin\frac{m\pi x}{a} \sin\frac{m\pi x'}{a} e^{izm^2\pi^2/2ka^2} \tag{7}$$

If we define $z_{11} = 4ka^2/\pi$, then

$$G_0(x,x';z) = \frac{1}{2a} e^{-ikz} \sum_{m=-\infty}^{\infty} e^{2\pi i m^2 z/z_{11}} \left\{ e^{-im\pi(x-x')/a} - e^{-im\pi(x+x')/a} \right\} \tag{8}$$

Equation (8) is the basis for the so-called Fourier-and Fresnel-imaging properties of this waveguide [6]. It is easily seen from (8) that $\exp(ikz)G_0$ is a periodic function of z:

$$G_0(x,x';z + z_{11}) = G_0(x,x';z)e^{-ikz_{11}} \tag{9}$$

In particular, since (6) implies that $G(x,x';0)$ is equal to $\delta(x-x')$ for $0 < x,\ x' < a$, we have

$$G_0(x,x';\ nz_{11}) = \delta(x-x')e^{-iknz_{11}} \tag{10}$$

for any integer n, i.e., the input plane field is replicated at each of the Fourier image planes $z = nz_{11}$. This Fourier imaging property of waveguides was apparently first noticed by Rivlin and Shul'dyaev [7] and discussed at length by a number of authors [5]-[12].

An even more interesting occurrence shows up at $= z_{pq}$, where

$$z_{pq} = \frac{p}{q} z_{11} \tag{11}$$

and p and q are some positive integers. Let us consider the sum

$$Q_{pq}(x) = \sum_{m=-\infty}^{\infty} e^{-im\pi x/a + 2\pi i m^2 z_{pq}/z_{11}} \tag{12}$$

Letting $m = p\ell + r$, where r runs from 0 to p-1, and using the Poisson sum formula,

$$Q_{pq}(x) = \frac{2a}{p} \sum_{r=0}^{p-1} e^{-i\pi r x/a + 2\pi i q r^2/p} \sum_{n=-\infty}^{\infty} \delta(x - \frac{2na}{p}) \tag{13}$$

from (8), then we have

$$G_0(x,x';z_{pq}) = e^{-ikz_{pq}} \sum_{n=-\infty}^{\infty} c_n(p,q)[\delta(x-x' - \frac{2na}{p}) - \delta(x + x' - \frac{2na}{p})] \tag{14}$$

where the coefficients c_n are given by

$$c_n(p,q) = \frac{1}{p} \sum_{r=0}^{p-1} e^{2\pi i r(rq+n)/p} \tag{15}$$

For $p = 1$, only one of the delta-functions, $\delta(x-x')$, is nonzero in the range $0 < (x,x') < a$, and we recover the single Fourier image described before. If $p > 1$ on the other hand, more of the delta-functions in (14) may appear in this range. Each one contributes to (6) a replica of the input field which is shifted by some amount in the x-direction, and whose amplitude is $|c_n(p,q)|$ times that of the original image. Any terms arising from the terms $\delta(x+x' - 2na/p)$ are inverted as well. Images of this type have been called Fresnel images.

For an input function not symmetric with respect to the center of the guide $x = a/2$, we have depicted the various images along with their (complex) amplitudes in Fig. 2 for z_{21}, z_{31} and z_{41}.

Propagation of a Gaussian Beam

Consider the initial field distribution

$$E_y(x,0) = e^{-(x-x_0)^2/2w_0^2} \tag{16}$$

i.e., a Gaussian beam centered at x_0 with waist parameter w_0. Let $0 < x_0 < a$, and assume that the "tails" of the beam are negligible at the walls of the guide:

$$w_0 \ll x_0; \qquad w_0 \ll a - x_0$$

In addition, we also suppose that the beam is well collimated, $kw_0 \ll 1$. Under these conditions and the paraxial approximation, (6) can be replaced by

$$E_y(x,z) = \int_{-\infty}^{\infty} e^{-(x'-x_0)^2/2w_0^2} G_0(x,x';z)dx' \tag{17}$$

Equation (17) implies an input field at $z = 0$ which consists not only of the original beam (16), but also of an infinite series of "mirror images" of the original beam, reflected in the upper and lower walls of the guide. Only the (very small amplitude) tails of the mirror images are present within the waveguide ($0 \le x \le a$). We wish to evaluate (17) for arbitrary (not necessarily rational) values of z/z_{11}.

The value of (17) can be expressed in terms of Jacobian theta-functions [13] whose properties can be used to obtain the expression:

$$E_y(x,z) = \frac{w_0}{f(\Delta z_p)} e^{-ikz} \sum_{n=-\infty}^{\infty} c_n(p,q) \left\{ e^{-(x-x_0-2na/p)^2/2f^2(\Delta z_p)} - e^{-(x-x_0-2na/p)^2/2f^2(\Delta z_p)} \right\} \tag{18}$$

where $c_n(p,q)$ is given by (15). Equation (18) represents a string of Fresnel images at $z = z_{pq}$, which have each propagated an additional distance Δz_p (so that $z = z_{pq} + \Delta z_p$) and broadened as they would in free space. We minimize the broadening of the beam by restricting

$$-z_{11}/2p \le \Delta z_p \le z_{11}/2p \tag{19}$$

The "complex waist parameter" $f(\Delta z_p)$ is defined as

$$f^2(\Delta z) = w_0^2 - i\Delta z_p/k \tag{20}$$

The situation is illustrated in Fig. 3. As Δz_p increases, more and more of the "mirror image" beams contribute significantly to the field in $0 \le x \le a$. The integer p should be chosen to be on the order of a/w_0 to achieve maximum computational efficiency.

Numerical Results

In Figure 4, we have traced the progress of a Gaussian beam with $kw_0 = 110.3$ in a waveguide with $ka = 973.4$ for selected values of z/z_{11} between 0 and 1/8.

It is seen that an appropriate value for p in this case is no more than about 6. In Fig. 5, an exact modal and approximate field computation is shown for a larger axial distance. It can be seen that even for $z \simeq 250\, z_{11}$, the paraxial approximation holds up quite well. For typical optical waveguide parameters, this could correspond to a length of about 50 m or so.

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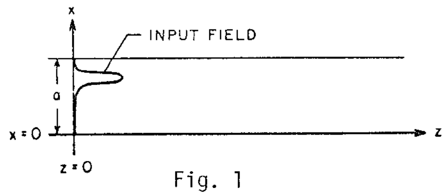

Fig. 1

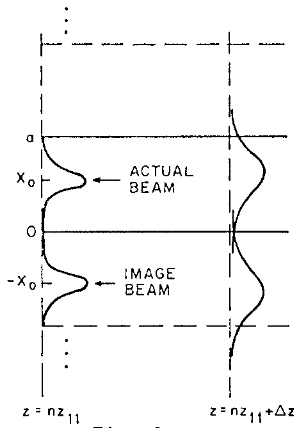

Fig. 3

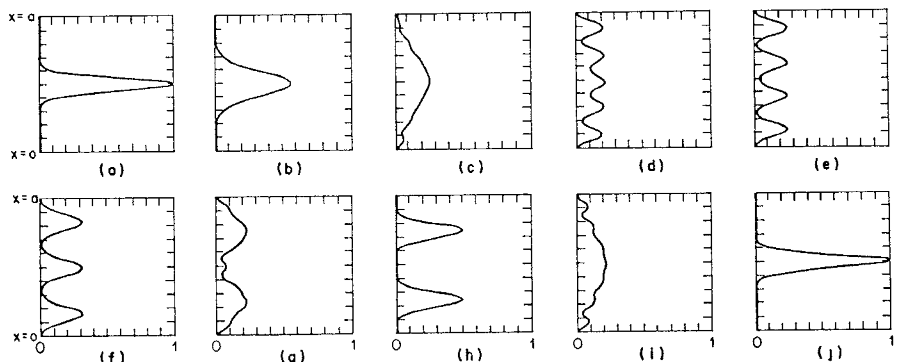

Fig. 4. Power distribution of Gaussian beam over 1/8-cycle: ka = 973.39; kw_o = 110.3. (a) z/z_{11} = 0; (b) 3/512; (c) 8/512; (d) 13/512/ (e) 16/512; (f) 22/512; (g) 28/512; (h) 32/512; (i) 55/512; (j) 64/512.

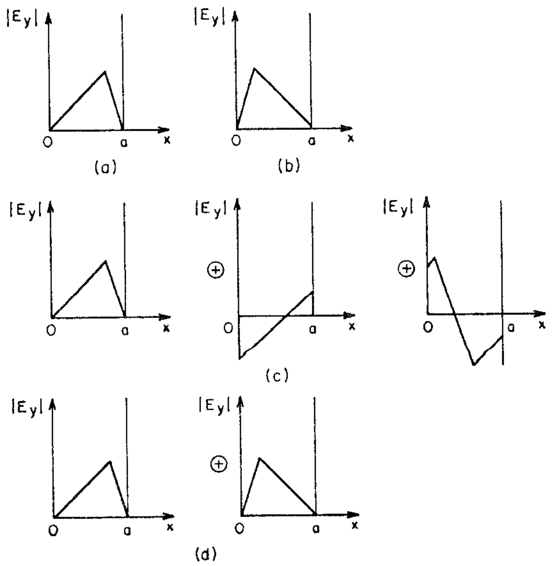

Fig. 2: Imaging of a nonsymmetrical field distribution. (a) $E_y(x,0)$. (b) $E_y(x,z_{21})$. (c) $E_y(x,z_{31})$. (d) $E_y(x,z_{41})$.

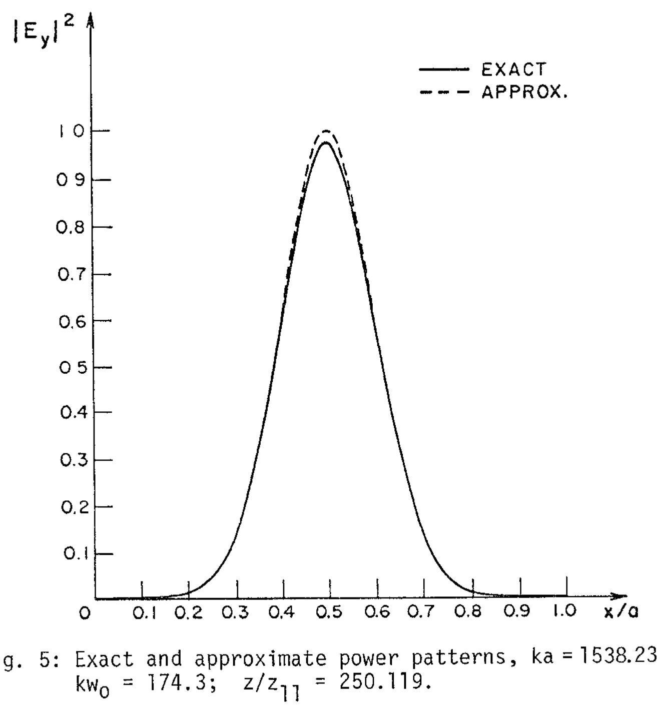

Fig. 5: Exact and approximate power patterns, ka = 1538.23; kw_0 = 174.3; z/z_{11} = 250.119.